On a sufficient condition that $\sqrt{s}$ is simply normal to base 2, for s not a perfect square

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Abstract

In [2] the author introduced a condition, Condition (TU), and proved that its validity implies the simple normality to base 2 of $\sqrt{s}$, for s not a perfect square. The argument also given in [2] that Condition (TU) is indeed valid was cumbersome. We give here a simpler direct proof that Condition (TU) is true. [1]

1 Introduction

In [2] the author introduced a condition called Condition (TU) and proved that it implied the simple normality to base 2 of $\sqrt{s}$ for s not a perfect square. Also given was an argument that Condition (TU) is true. This argument was unnecessarily long, and was hard to follow according to some readers. Recently I have found a simpler proof of the validity of Condition (TU); it is presented in Theorem 1.

Consider numbers ω in the unit interval, and represent the dyadic expansion of ω as

$$\omega = .x_1 x_2 \cdots, \qquad x_i = 0 \text{ or } 1. \tag{1}$$

Also of interest is the dyadic expansion of $\nu = \omega^2$:

$$\nu = \omega^2 = .u_1 u_2 \cdots, \qquad u_i = 0 \text{ or } 1. \tag{2}$$

Throughout this paper it will be assumed that ν is irrational. Then ω is also irrational and both expansions are uniquely defined. It will be convenient to refer to the expansion of ω as an x sequence and the expansion of ν as a u sequence. A point of the unit interval can also be denoted by its coordinate

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representation, that is, $\omega = (x_1, x_2, \cdots)$ or $\nu = (u_1, u_2, \cdots)$. The coordinate functions $X_n(\omega) = x_n$ and $U_n(\nu) = u_n$ give the nth coordinates of ω and ν respectively.

Given any dyadic expansion $.s_1 s_2 \cdots$ and any positive integer n, the sequence of digits $s_n, s_{n+1}, \cdots$ is called a *tail* of the expansion. Two expansions are said to have the same tail if there exists n so large that the tails of the sequences from the nth digit are equal.

The average

$$f_n(\omega) = \frac{x_1 + x_2 + \cdots + x_n}{n} \tag{3}$$

is the relative frequency of 1's in the first n digits of the expansion of ω. Simple normality for ω is the assertion that $f_n(\omega) \to 1/2$ as n tends to infinity. Let n_k be any fixed subsequence and define

$$f(\omega) = \limsup_{k \to \infty} f_{n_k}(\omega). \tag{4}$$

We note that the function f is a tail function with respect to the x sequence, that is, $f(\omega)$ is determined by any tail $x_n, x_{n+1}, \cdots$ of the coordinates of ω.[1]

We now observe that the average f_n, defined in terms of the x sequence, can also be expressed as a function $h_n(\nu)$ of the u sequence because the x and u sequences uniquely determine each other. This relationship has the simple form $f_n(\omega) = f_n(\sqrt{\nu}) = h_n(\nu)$. Define $h(\nu) = \limsup_k h_{n_k}(\nu)$; then clearly $f(\omega) = h(\nu)$.

Definition: Let f be defined as in relation 4 for any fixed subsequence n_k. We say that Condition (TU) is satisfied if $f(\omega) = h(\nu)$ is a tail function *with respect to the u sequence* whatever the sequence n_k, that is, for any ω and any positive integer n, $f(\omega)$ only depends on $u_n, u_{n+1}, \cdots$, the tail of the expansion of $\nu = \omega^2$. (The notation "TU" is meant to suggest the phrase "tail with respect to the u sequence".)

An immediate consequence of Condition (TU) is:

Proposition 1 *Let η be the dyadic expansion of an irrational number. Let η_1 be a dyadic expansion that agrees with η at all but a finite number of indices. If Condition (TU) is satisfied then*

$$\lim_n (f_n(\sqrt{\eta}) - f_n(\sqrt{\eta_1})) = 0.$$

[1] In fact, f satisfies a more stringent requirement: it is an invariant function (with respect to the x sequence) in the following sense: let T be the 1-step shift transformation on Ω to itself given by

$$T(.x_1 x_2 \cdots) = .x_2 x_3 \cdots.$$

A function g on Ω is invariant if $g(T\omega) = g(\omega)$ for all ω. Any invariant function is a tail function.

2 Proof of Condition (TU)

Lemma 1 *Let $\omega^2 = \nu$.*
(a) Let $u_1, u_2, \cdots, u_r$ be the initial segment of length r of ν. Then there exists a positive integer $N = N(\omega, r)$ such that each u_i, $i \leq r$ is a function of $x_1, x_2, \cdots, x_N$.
(b) Let $x_1, x_2, \cdots, x_n$ be the initial segment of length n of ω. Then there exists a positive integer $m = m(\nu, n)$ such that each x_j, $j \leq n$ is a function of $u_1, u_2, \cdots, u_m$.

The proof can be found in [2], lemmas 2 and 3.

The following arguments will use some elementary ideas from the calculus of finite differences. An introduction to these ideas may be found, for example, in [1]. We review some of the notation. Let $v(y_1, \cdots, y_l) = v(\boldsymbol{y})$ be a function on the l-fold product space S^l where the $y_i \in S$, a set of real numbers. Suppose that the variable y_i is changed by the amount Δy_i such that the l-tuple $\boldsymbol{y^{(1)}} = (y_1, \cdots, y_l)$ is taken into $\boldsymbol{y^{(2)}} = (y_1 + \Delta y_1, \cdots, y_l + \Delta y_l)$ in the domain of definition of v. Put $v(\boldsymbol{y^{(2)}}) - v(\boldsymbol{y^{(1)}}) = \Delta v$, and let

$$\begin{aligned} \Delta v_i &= v(y_1, \cdots, y_{i-1}, y_i + \Delta y_i, y_{i+1} + \Delta y_{i+1,\cdots}, y_l + \Delta y_l) \\ &- v(y_1, \cdots, y_{i-1}, y_i, y_{i+1} + \Delta y_{i+1,\cdots}, y_l + \Delta y_l). \end{aligned} \tag{5}$$

Then $\Delta v = \sum_i \Delta v_i$ is the total change in v induced by changing all of the y_i, where this total change is written as a sum of step-by-step changes in the individual y_i. Formally, by dividing, we can write

$$\Delta v = \sum_i (\Delta v_i / \Delta y_i) \cdot \Delta y_i. \tag{6}$$

If some $\Delta y_{i_0} = 0$, its coefficient in relation 6 has the form 0/0. No matter how the coefficient is defined in this case the contribution of the i_0 term to Δv is 0. For our purposes it is convenient to define the coefficient to be Δv_{i_0} evaluated as though y_{i_0} were equal to 0 and Δy_{i_0} were equal to 1.

Let us then formally define the *partial difference of v with respect to y_i, evaluated at the pair $(\boldsymbol{y^{(1)}}, \boldsymbol{y^{(2)}})$* by

$$\begin{aligned} \frac{\Delta v}{\Delta y_i} &= \Delta v_i / \Delta y_i, \quad \text{if } \Delta y_i \neq 0, \\ &= \Delta v_i \text{ evaluated as though } y_i = 0 \text{ and } \Delta y_i = 1, \quad \text{if } \Delta y_i = 0. \end{aligned} \tag{7}$$

Notice that the forward slash (/) in this relation expresses division and the horizontal slash on the left hand side is the partial difference operator.

The sum Δv of relation 6 is called the *total difference of* v evaluated at the given pair and can now be written

$$\Delta v = \sum_i \frac{\Delta v}{\Delta y_i} \cdot \Delta y_i. \tag{8}$$

The ith summand in relation 8 is called the *ith partial difference* of v relative to the given pair. The partial and total differences are the discrete analogs of the partial and total differentials in the theory of differentiable functions of several real variables and the partial difference with respect to a given y variable is the analog of the partial derivative. The ith partial difference of v at a given pair is a measure of the contribution of Δy_i to Δv when all the other y variables are held constant.

Returning to our particular problem, we say that ω and $\nu = \omega^2$ are points (or expansions) that *correspond* to one another. As seen in Section 1 the average $f_n(\omega)$ of relation 3 can be written as a function $h_n(\nu)$. With a slight abuse of notation we can write

$$f_n(x_1, \cdots, x_n) = f_n(\omega) = h_n(\nu) = h_n(u_1, u_2, \cdots). \tag{9}$$

Fix a point ω with corresponding point ν, and for each x_j let Δx_j be a given increment chosen independently ($\Delta x_j = 0, 1$, or -1). Let $\omega^{(1)}$ have coordinates $x_j + \Delta x_j$ and let $\nu^{(1)}$ correspond to $\omega^{(1)}$. Let the ith coordinate of $\nu^{(1)}$ be $u_i + \Delta u_i$. Thus the changes Δx_j in the x coordinates have induced changes Δu_i in the u coordinates. Of course this process could have been reversed: independent changes in the u's induce changes in the x's.

The following two lemmas are finite difference analogs of the total differential formulas in the theory of differentiable functions of a function of several variables. The first result is fairly evident.

Lemma 2 *At the pair $(\omega, \omega^{(1)})$, Δf_n can be represented as a total difference*

$$\begin{aligned}\Delta f_n &= f_n(\omega^{(1)}) - f_n(\omega) = \\ &\quad f_n(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots x_n + \Delta x_n) - f_n(x_1, x_2, \cdots x_n) \\ &= \frac{1}{n} \sum_{1 \le j \le n} \Delta x_j\end{aligned} \tag{10}$$

Proof: Decompose according to the recipe given in relations 5 to 8 to get [1]

$$\frac{\Delta f_n}{\Delta x_j} = \frac{1}{n}, \quad j \le n \text{ and } = 0, \quad j > n.$$

[1] Our definitions require the "denominator" of a partial difference to be a variable, so strictly speaking x_j in this relation should be replaced by X_j, the jth coordinate variable, with an added notation that it is evaluated at the given base point ω. The present notation is simpler and will be followed throughout.

The next lemma is more interesting.

Lemma 3 *At the pair $(\nu, \nu^{(1)})$, Δh_n can be represented as a total difference*

$$\begin{aligned} \Delta h_n &= h_n(\nu^{(1)}) - h_n(\nu) = h_n(u_1 + \Delta u_1, u_2 + \Delta u_2, \cdots) - h_n(u_1, u_2, \cdots) \\ &= \sum_{i \geq 1} \frac{\Delta h_n}{\Delta u_i} \Delta u_i = \sum_{i \geq 1} (\Delta h_{n,i} / \Delta u_i) \, \Delta u_i, \quad \Delta u_i \neq 0 \end{aligned} \tag{11}$$

where

$$\begin{aligned} \Delta h_{n,i} = \Delta h_{n,i}(\nu, \nu^{(1)}) = \\ h_n(u_1, \cdots, u_{i-1}, u_i + \Delta u_i, u_{i+1} + \Delta u_{i+1}, \ldots) - \\ h_n(u_1, \cdots, u_{i-1}, u_i, u_{i+1} + \Delta u_{i+1}, \ldots) \quad . \end{aligned} \tag{12}$$

The formally infinite sum of relation 11 reduces to a finite sum. More precisely, given the pair $(\nu, \nu^{(1)})$, there exists an integer m such that the partial differences $\Delta h_{n,i} / \Delta u_i = 0$ for all $i > m$. The number of non-vanishing terms in the sum depends on ν and n.

Proof: The recipe given in relation 11 for decomposing Δh_n is given by the definitions stated in relations 5 through 8. The pair $(\nu, \nu^{(1)})$ corresponds to the pair $(\omega, \omega^{(1)})$. To see that the sum in relation 11 is finite, note that the function $h_n = f_n$ only depends on $x_1, x_2, \cdots x_n$. Given ν, Lemma 1 proves the existence of an integer m such that for all $i > m$

$$\frac{\Delta x_j}{\Delta u_i} = 0, \qquad 1 \leq j \leq n.$$

and therefore the terms $\Delta h_{n,i}$ of relation 12 are 0 for $i > m$. Thus the terms in the sum of relation 11 vanish for $i > m$ and the formula of relation 11 represents a finite sum. This concludes the proof of the lemma.

We have seen (lemma 2) that the partial difference of f_n with respect to any fixed x_j is $1/n$. This means that the contribution to changes in the averages f_n of any change in a single x_j tends to 0. But how about the partial differences of h_n with respect to a single fixed u_i? Does a change in u_i induce changes in h_n that die out in the limit? The answer is not obvious. Let u_r be a fixed u variable. Relation 12 shows that

$$\Delta h_{n,r} = \frac{1}{n} \sum_{1 \leq j \leq n} \Delta x'_j \tag{13}$$

for some sequence of changes of x variables (see proof of theorem 1 below). If $\Delta u_r \neq 0$ the partial difference of h_n with respect to u_r just differs from $\Delta h_{n,r}$

by a factor of ± 1. Therefore the questions posed above reduce to asking what the limit points of the right hand side of relation 13 are. Heuristic considerations suggest why we might expect the averages in relation 13 to converge to 0: h_n depends on larger and larger initial segments of u variables as n increases and a certain symmetry exists in the problem. It seems reasonable to suspect that change in a single u variable is not going to have much of an effect on h_n for large n. We now set out to prove that this suspicion is true.

Since the u variables are functions of the x variables and vice versa, it is possible to consider either set of variables independent and the other set dependent on them. We choose to take the x variables independent. The power of this approach becomes apparent in the next result which solves our problem.

Theorem 1 *Assume that the u variables are functions of independent x variables. Then*
(a): For all i, the partial differences of h_n with respect to u_i in relation 11 satisfy

$$\lim_n \frac{\Delta h_n}{\Delta u_i} = 0. \tag{14}$$

(b): Condition (TU) is true.

Proof of (a): The ith partial differences Δu_i referenced in (a) are all non-zero, so let r be a fixed positive integer with $\Delta u_r \neq 0$. Consider relation 12. The right hand side expresses $\Delta h_{n,r}$ as the difference $h_n(\nu_2) - h_n(\nu_1)$ evaluated at the two points

$$\begin{aligned} \nu_2 &= (u_1, \cdots, u_{r-1}, u_r + \Delta u_r, u_{r+1} + \Delta u_{r+1}, \ldots) \text{ and} \\ \nu_1 &= (u_1, \cdots, u_{r-1}, u_r, u_{r+1} + \Delta u_{r+1}, \ldots) \end{aligned} \tag{15}$$

The irrationality of $\nu^{(1)}$ implies that ν_1 and ν_2 are also irrational. For $k = 1, 2$, let ν_k correspond to $\omega_k = (x_{(1,k)}, x_{(2,k)}, \cdots)$ and put $x_{(j,2)} - x_{(j,1)} = \Delta x'_j$. Let the differences in the u coordinates at (ν_1, ν_2) be denoted by $\Delta u'_i$. Then $\Delta u'_i = 0$ for $i \neq r$, $\Delta u'_r = \Delta u_r$. We study the functions f_n and h_n at the pairs (ω_1, ω_2) and (ν_1, ν_2) respectively. At the pairs (ω_1, ω_2) and (ν_1, ν_2), lemmas 2 and 3 correspond to

$$\Delta h'_n = \Delta f'_n = f_n(\omega_2) - f_n(\omega_1) = \frac{1}{n} \sum_{1 \le j \le n} \Delta x'_j \tag{16}$$

and

$$\Delta h'_n = h_n(\nu_2) - h_n(\nu_1) = \frac{\Delta h'_n}{\Delta u_r} \Delta u_r, \text{ where } \quad \frac{\Delta h'_n}{\Delta u_r} = \pm \frac{1}{n} \sum_{1 \le j \le n} \Delta x'_j. \tag{17}$$

At the pair (ν_1, ν_2)

$$\Delta h_n' = \frac{\Delta h_n'}{\Delta u_r}\Delta u_r = h_n(\nu_2) - h_n(\nu_1) = \frac{\Delta h_n}{\Delta u_r}\Delta u_r = \Delta h_{n,r}$$

so that

$$\frac{\Delta h_n'}{\Delta u_r} = \frac{\Delta h_n}{\Delta u_r} \tag{18}$$

and

$$\Delta h_n' = \frac{\Delta h_n}{\Delta u_r}\,\Delta u_r. \tag{19}$$

Since u_r is a function of the x variables, at the pair (ω_1, ω_2) relations 5 through 8 give the representation

$$\Delta u_r = \sum_{j\geq 1} \frac{\Delta u_r}{\Delta x_j'}\,\Delta x_j'.$$

By lemma 1 there exists $N = N(\omega_1, r)$ such that the changes $\Delta x_j'$, $j > N$ cause no change in u_r, that is,

$$\frac{\Delta u_r}{\Delta x_j'} = 0, \qquad j > N.$$

It follows that there is the finite decomposition

$$\Delta u_r = \sum_{1\leq j\leq N} \frac{\Delta u_r}{\Delta x_j'}\,\Delta x_j'. \tag{20}$$

Using relation 20, relation 19 can be rewritten

$$\Delta h_n' = \sum_{1\leq j\leq N} \frac{\Delta h_n}{\Delta u_r}\frac{\Delta u_r}{\Delta x_j'}\,\Delta x_j'. \tag{21}$$

Let n_k be any subsequence for which there is convergence in relation 21, that is,

$$\lim_k \Delta h_{n_k}' = \Delta h' \text{ and } \lim_k \frac{\Delta h_{n_k}}{\Delta u_r} = \frac{\Delta h}{\Delta u_r}$$

where the right hand sides are defined by the existing limits. Then

$$\Delta h' = \sum_{1\leq j\leq N} \frac{\Delta h}{\Delta u_r}\frac{\Delta u_r}{\Delta x_j'}\,\Delta x_j'. \tag{22}$$

Let $p \le N$ be an index with

$$\frac{\Delta u_r}{\Delta x'_p}\,\Delta x'_p \neq 0. \tag{23}$$

Such p exists by relation 20 since $\Delta u_r \neq 0$. Now observe that $\Delta h' = \lim_k \Delta h'_{n_k}$ is a tail function considered as a function of the $\Delta x'_j$ variables (see relation 16), so is not a function of $\Delta x'_j$ for any fixed j. Moreover,

$$\frac{\Delta h}{\Delta u_r} \text{ is a tail function with respect to the } \Delta x'_j \text{ (see relations 17 and 18).} \tag{24}$$

Take the partial difference with respect to the pth coordinate variable on both sides of relation 22. By the tail property of $\Delta h'$ stated above,

$$\frac{\Delta h'}{\Delta x'_p} = 0. \tag{25}$$

Independence of the x variables implies

$$\frac{\Delta x'_j}{\Delta x'_p} = 0, \qquad j \neq p.$$

Use relations 23 and 24 and the foregoing relation to see that the partial difference with respect to the pth coordinate variable on the right hand side of relation 22 can be written

$$\sum_{1\le j\le N} \frac{\Delta h}{\Delta u_r}\frac{\Delta u_r}{\Delta x'_j}\,\frac{\Delta x'_j}{\Delta x'_p} = \frac{\Delta h}{\Delta u_r}\frac{\Delta u_r}{\Delta x'_p} = \pm\frac{\Delta h}{\Delta u_r} \tag{26}$$

and so from relations 25 and 22 relation 26 implies

$$\frac{\Delta h}{\Delta u_r} = 0. \tag{27}$$

The subsequence n_k is associated with an arbitrary limit point so the above argument shows this limit point is unique, that is

$$\lim_n \frac{\Delta h_n}{\Delta u_r} = 0$$

and this proves (a).
Proof of (b): Given any subsequence n_k and any positive integer M, part (a) of this theorem proves that in relation 11

$$\limsup_k \Delta h_{n_k} = \limsup_k \left(\sum_{i>M} \frac{\Delta h_{n_k}}{\Delta u_i}\,\Delta u_i \right).$$

The relation shows that $\limsup_k \Delta h_{n_k}$ does not depend on the differences of any initial segment of u coordinates for the given pair in relation 11. Since lemma 3 makes no restrictions on pairs (other than they are well defined), this assertion is true for all meaningful pairs. This implies that $\limsup_k h_{n_k} = \limsup_k f_{n_k}$ is a tail function with respect to the u variables, that is, Condition (TU) is true.

References

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